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Research Article

Impulsive Integrodifferential Equations Involving Nonlocal Initial Conditions

Rong-Nian Wang and Jun Xia

Department of Mathematics, NanChang University, NanChang, JiangXi 330031, China

Correspondence should be addressed to Rong-Nian Wang, rnwang@mail.ustc.edu.cn

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We focus on a Cauchy problem for impulsive integrodifferential equations involving nonlocal initial conditions, where the linear part is a generator of a solution operator on a complex Banach space. A suitable mild solution for the Cauchy problem is introduced. The existence and uniqueness of mild solutions for the Cauchy problem, under various criterions, are proved. In the last part of the paper, we construct an example to illustrate the feasibility of our results.

1. Introduction

Let $(X, \|\cdot\|)$ denote a complex Banach space and denote $\mathcal{L}(X)$ by the space of all bounded linear operators from X into X with the usual operator norm $\|\cdot\|_{\mathcal{L}(X)}$. Let us recall the following definitions.

Definition 1.1 (see [1]). Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function and $\gamma \geq 1$. Then the expression

$$(I^\gamma f)(t) = \int_0^t \frac{(t-s)^{\gamma-2}}{\Gamma(\gamma-1)} f(s)ds \tag{1.1}$$

is called the Riemann-Liouville integral of order $\gamma - 1$.

Definition 1.2 (see [2]). Let A be a linear and closed operator with domain $D(A)$ defined on X. By a solution operator associated with A in X, we mean a family $\{Q_\alpha : \mathbb{R}^+ \to \mathcal{L}(X)\}$ of strongly continuous operators satisfying

(1) $\{\lambda^\alpha : \operatorname{Re}\lambda > \theta\} \subset \rho(A)$ and

(2)

$$\lambda^{\alpha-1}\mathcal{R}(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t} Q_\alpha(t)x\,dt \quad (\operatorname{Re}\lambda > \theta,\ x \in X), \tag{1.2}$$

where $\theta \in \mathbb{R}$ is a constant and $\mathcal{R}(\lambda, A) = (\lambda I - A)^{-1}$ stands for the resolvent of A. In this case, we also say that $Q_\alpha(t)$ is a solution operator generated by A.

Remark 1.3. It is to be noted that in the border case $\alpha = 1$, the family $Q_\alpha(t)$ corresponds to a classical strongly continuous semigroup, whereas in the case $\alpha = 2$ a solution operator corresponds to the concept of a cosine family. Moreover, according to [3], one can find that solution operators are a particular case of (a, k)-regularized families and a solution operator $Q_\alpha(t)$ corresponds to a $(1, t^{\alpha-1}/\Gamma(\alpha))$-regularized family.

Remark 1.4. Note that solution operator $Q_\alpha(t)$ does not satisfy the semigroup property.

Remark 1.5. Various solution operators are usually key tools in dealing with the abstract Cauchy problems and related issues. For more information, please see, for example, [4–11] and references therein.

Starting from some speculations of Leibniz and Euler, the fractional calculus (such as the Riemann-Liouville fractional integral) which allows us to consider integration and differentiation of any order, not necessarily integer, have been the object of extensive study for analyzing not only stochastic processes driven by fractional Brownian motion, but also nonrandom fractional phenomena in physics and optimal control (cf. e.g., [1, 12, 13]). One of the emerging branches of the study is the Cauchy problems of abstract differential equations involving fractional integration or fractional differentiation (see, e.g., [1, 14–17]). Let us point out that many phenomena in engineering, physics, economy, chemistry, aerodynamics, and electrodynamics of complex medium can be modeled by this class of equations.

In the present paper we study the existence and uniqueness of mild solutions for the Cauchy problem for impulsive integrodifferential equations involving nonlocal initial conditions in the form

$$\begin{gathered} u'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds = F(t, u(t)), \quad 0 \le t \le a,\ t \ne t_i, \\ u(0) = H(u), \\ u(t_i^+) = u(t_i^-) + T_i(u(t_i^-)), \quad i = 1, \dots, n, \end{gathered} \tag{1.3}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a generator of a solution operator $Q_\alpha(t)$, $0 < t_1 < \cdots < t_n < a$, $u(t_i^+) = \lim_{\delta\to 0^+} u(t_i + \delta)$ and $u(t_i^-) = \lim_{\delta\to 0^-} u(t_i + \delta)$ stand for the right and left limits of $u(t)$ at $t = t_i$, respectively, and $F : [0, a] \times X \to X$, $T_i : X \to X$, $i = 1, \dots, n$ are appropriate functions to be specified later. As can be seen, the convolution integral in (1.3) is the Riemann-Liouville fractional integral, and the function H constitutes a nonlocal condition.

As usual, the solution $t \to u(t)$ with the points of discontinuity at the moments t_i $(i = 1,\dots,n)$ follows that $u(t_i) = u(t_i^-)$, that is, at which it is continuous from the left.

We mention that in recent years, the theory of various integrodifferential equations in Banach spaces has been studied deeply due to their important values in sciences and technologies, and many significant results have been established (see, e.g., [2, 18–23] and references therein).

Interest in impulsive nonlocal Cauchy problems stems mainly from the observation that on one side, nonlocal initial conditions have better effects in treating physical problems than the usual ones (see [21, 22, 24–27] and the references therein for more detailed information about the importance of nonlocal initial conditions in applications); on the other side, the dynamics of many evolutionary processes from some research fields are subject to abrupt changes of states at certain moments of time between intervals of continuous evolution, such changes can be well approximated as being instantaneous changes as state, that is, in the form of "impulses" (cf. [20, 28] and the references therein). This class of equations has been the object of extensive study in recent years, see [29–31] and the references therein for more comments and citations. It is worth mentioning that in [31], Liang et al. considered the following impulsive nonlocal Cauchy problem

$$\begin{gathered} u'(t) = Au(s) + f(t,u(t)), \quad 0 \le t \le a,\ t \ne t_i, \\ u(0) + g(u) = u_0, \\ u(t_i^+) - u(t_i^-) = I_i(u(t_i)), \quad i = 1,\dots,n,\ 0 < t_1 < \cdots < t_n < a, \end{gathered} \tag{1.4}$$

where A is the generator of a strongly continuous semigroup in a Banach space and the existence and uniqueness of mild and classical solutions for the Cauchy problem, under various criterions, are proved. Also, Wang et al. [32] proved the existence and uniqueness of mild and classical solutions for the nonlocal Cauchy problem in the form

$$\begin{gathered} u'(t) = Au(s) + h(t,u(t)), \quad t > 0, \\ u(0) + H(t_1,\dots,t_p,u) = u_0, \end{gathered} \tag{1.5}$$

where $0 < t_1 < \cdots < t_{p-1} < t_p < \infty$ $(p \in N)$, A is a ω-almost sectorial operator (not necessarily densely defined).

In this work, motivated by the above contributions, we shall combine these earlier work and extend the study to the Cauchy problem (1.3). New existence and uniqueness results in the case when A is a generator of a solution operator, under various criterions, are proved. In the last part of paper, we construct an example to illustrate the feasibility of our results.

2. Preliminaries

Throughout this paper, we take $C([0,a];X)$ to be the Banach space of all X-valued continuous functions from $[0,a]$ into X endowed with the uniform norm topology

$$\|u\|_a = \sup\{\|u(t)\|;\ t \in [0,a]\}. \tag{2.1}$$

Put

$$I_0 = [0, t_1], \qquad I_i = (t_i, t_{i+1}], \quad i = 1, \dots, n, \tag{2.2}$$

with $t_0 = 0$, $t_{n+1} = a$, and let u_i be the restriction of a function u to I_i $(i = 0, 1, \dots, n)$.

Consider the set of functions

$$\begin{aligned} \mathrm{PC}([0,a];X) = \{&u : [0,a] \longrightarrow X;\ u_i \in C(I_i;X),\ i = 0, 1, \dots, n, \\ &u(t_i^+), u(t_i^-) \text{ exist, and satisfy } u(t_i) = u(t_i^-) \text{ for } i = 1, \dots, n\}, \end{aligned} \tag{2.3}$$

endowed with the norm

$$\|u\|_{\mathrm{PC}} = \max\left\{ \sup_{t \in I_i} \|u_i(t)\|;\ i = 0, 1, \dots, n \right\}. \tag{2.4}$$

It is easy to see $\mathrm{PC}([0,a];X)$ is a Banach space.

Let $1 < \alpha < 2$. It follows from [33] that if A is sectorial of type θ $(\in \mathbb{R})$, that is, A is a closed linear operator, and there exist constants $\varphi \in (0, \pi/2)$ and $C' > 0$ such that $\mathbb{C} - \{\theta + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \varphi\} \subset \rho(A)$ and

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{C'}{|\lambda - \theta|}, \quad \lambda \in \mathbb{C} - \{\theta + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \varphi\}, \tag{2.5}$$

then A is a generator of a solution operator $Q_\alpha(t)$, which is given by

$$Q_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha - 1} (\lambda^\alpha - A)^{-1} d\lambda, \tag{2.6}$$

provided that $0 \le \varphi < (1 - \alpha/2)\pi$, where Γ is a suitable path lying outside the sector $\{\theta + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \varphi\}$. And Cuesta [18, Theorem 1], has proved that if A is a sectorial operator of type $\theta < 0$ and there is a positive constant C_α which depends on C' such that the estimate

$$\|Q_\alpha(t)\|_{\mathcal{L}(X)} \le \frac{C_\alpha}{1 + |\theta| t^\alpha} \tag{2.7}$$

holds for all $t \ge 0$.

We recall that the Laplace transform of a abstract function $g \in L^1(\mathbb{R}^+, X)$ is defined by

$$\widehat{g}(\zeta) := \int_0^\infty e^{-\zeta t} g(t) dt. \tag{2.8}$$

We first treat the following problem:

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds + f(t), \quad t > 0, \ 1 < \alpha < 2, \\ u(0) = u_0. \tag{2.9}$$

Formally applying the Laplace transform in (2.9), we obtain

$$\lambda \widehat{u}(\zeta) - u_0 = \lambda^{1-\alpha} A \widehat{u}(\zeta) + \widehat{f}(\lambda), \tag{2.10}$$

which establishes the following result:

$$\widehat{u}(\zeta) = \lambda^{\alpha-1} \mathcal{R}(\lambda^\alpha, A) u_0 + \lambda^{\alpha-1} \mathcal{R}(\lambda^\alpha, A) \widehat{f}(\lambda). \tag{2.11}$$

This means that

$$u(t) = Q_\alpha(t) u_0 + \int_0^t Q_\alpha(t-s) f(s) ds. \tag{2.12}$$

Motivated by the above consideration, we give the following definition.

Definition 2.1. Let $1 < \alpha < 2$. A solution $u \in C([0,a]; X)$ of the integral equation

$$u(t) = Q_\alpha(t) H(u) + \int_0^t Q_\alpha(t-s) F(s, u(s)) ds, \quad t \in [0,a], \tag{2.13}$$

is called a mild solution of the following problem:

$$u'(t) - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)ds = F(t, u(t)), \quad t \in [0,a], \\ u(0) = H(u), \tag{2.14}$$

where Q_α is the solution operator generated by A.

We list the following basic assumptions of this paper.

(H_1) $F : [0,a] \times X \to X$ is continuous in t on $[0,a]$ and there exists a constant $L_F > 0$ such that

$$\|F(t, u_1) - F(t, u_2)\| \le L_F \|u_1 - u_2\| \tag{2.15}$$

for all $(t, u_1), (t, u_2) \in [0,a] \times X$.

(H_1') $F : [0,a] \times X \to X$ is continuous and there exists a function $\rho(t) \in L^1([0,a];\mathbb{R}^+)$ such that

$$\|F(t,u_1) - F(t,u_2)\| \le \rho(t)\|u_1 - u_2\| \tag{2.16}$$

for all $t \in [0,a]$, $u_1, u_2 \in X$.

(H_2) $H : \mathrm{PC}([0,a];X) \to X$ is completely continuous and there exists a continuous nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $r > 0$,

$$\begin{gathered} \sup_{\|u\|_{\mathrm{PC}} \le r} \|H(u)\| \le \Phi(r), \\ \liminf_{r \to +\infty} \frac{\Phi(r)}{r} = \eta < +\infty. \end{gathered} \tag{2.17}$$

(H_2') $H : \mathrm{PC}([0,a];X) \to X$ is Lipschitz continuous with Lipschitz constant L_H.

(H_3) For $i = 1,\dots,n$, $T_i : X \to X$ is Lipschitz continuous with Lipschitz constant L_i.

(H_3') For $i = 1,\dots,n$, $T_i : X \to X$ is completely continuous and there exists a continuous nondecreasing function $\Upsilon_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $r > 0$,

$$\sup_{\|u\| \le r} \|T_i(u)\| \le \Upsilon_i(r), \qquad \liminf_{r \to +\infty} \frac{\Upsilon_i(r)}{r} = \lambda_i < +\infty. \tag{2.18}$$

The following fixed-point theorem plays a key role in the proof of our main results.

Lemma 2.2 (see [34]). *Let Y be a convex, bounded, and closed subset of a Banach space X and let $\Psi : Y \to Y$ be a condensing map. Then, Ψ has a fixed point in Y.*

3. Main Results

To set the framework for our main existence results, we will make use of the following lemma.

Lemma 3.1. *Let $1 < \alpha < 2$. Assume that A is a sectorial operator of type $\theta < 0$ and $Q_\alpha(t)$ is a solution operator generated by A. Suppose in addition that $F : [0,a] \times X \to X$ is a continuous function. If $u \in C([0,a];X)$ is a mild solution of the Cauchy problem* (2.14) *in the sense of Definition 2.1, then, u satisfying the following impulsive integral equation:*

$$\begin{aligned} u(t) = {} & \Phi_i^\alpha(t) H(u) + \int_{t_i}^{t} Q_\alpha(t-s) F(s,u(s))\,ds \\ & + \sum_{1 \le j \le i} \int_{t_{j-1}}^{t_j} \Phi_{i,j}^\alpha(t) Q_\alpha(t_j - s) F(s,u(s))\,ds \\ & + \sum_{1 \le j \le i} \Phi_{i,j}^\alpha(t) T_j\Big(u\Big(t_j^-\Big)\Big), \quad t \in I_i,\ i = 0,1,\dots,n, \end{aligned} \tag{3.1}$$

is a mild solution of problem (1.3)*, where*

$$\Phi_i^{\alpha}(t) := \begin{cases} Q_\alpha(t) & \text{if } i = 0, \\ \displaystyle\prod_{1\le j\le i} Q_\alpha(t-t_i)Q_\alpha\left(t_j - t_{j-1}\right) & \text{if } i \ge 1, \end{cases} \tag{3.2}$$

$$\Phi_{i,j}^{\alpha}(t) := \begin{cases} Q_\alpha(t-t_i) & \text{if } j = i, \\ \displaystyle\prod_{j<k\le i} Q_\alpha(t-t_i)Q_\alpha(t_k - t_{k-1}) & \text{if } j < i. \end{cases} \tag{3.3}$$

Proof. Assume that $u \in C([0,a];X)$ is a mild solution of (2.14) in the sense of Definition 2.1. Obviously, if $t \in I_0$, then one sees from Definition 2.1, that the assertion of theorem remains true. Thus, the rest proof of the theorem is done under $t \in I_i$ $(i = 1,\dots,n)$.

By Definition 2.1, note that

$$u(t) = Q_\alpha(t)H(u) + \int_0^t Q_\alpha(t-s)F(s,u(s))ds \tag{3.4}$$

for all $t \in I_0$. Taking $t = t_1$, then we get

$$u(t_1) = Q_\alpha(t_1)H(u) + \int_0^{t_1} Q_\alpha(t_1-s)F(s,u(s))ds. \tag{3.5}$$

Hence, it follows form $u(t_1^+) = u(t_1^-) + T_1(u(t_1^-))$ that

$$u\left(t_1^+\right) = Q_\alpha(t_1)H(u) + \int_0^{t_1} Q_\alpha(t_1-s)F(s,u(s))ds + T_1\left(u\left(t_1^-\right)\right). \tag{3.6}$$

If $t \in I_1$, then combining Definition 2.1 and the result above, we deduce that

$$\begin{aligned} u(t) &= Q_\alpha(t-t_1)u\left(t_1^+\right) + \int_{t_1}^t Q_\alpha(t-s)F(s,u(s))ds \\ &= Q_\alpha(t-t_1)Q_\alpha(t_1)H(u) + Q_\alpha(t-t_1)T_1\left(u\left(t_1^-\right)\right) \\ &\quad + \int_0^{t_1} Q_\alpha(t-t_1)Q_\alpha(t_1-s)F(s,u(s))ds \\ &\quad + \int_{t_1}^t Q_\alpha(t-s)F(s,u(s))ds. \end{aligned} \tag{3.7}$$

This proves, for the case $i = 1$, that the conclusion of theorem holds.

Now taking $t = t_2$ in (3.7), one has

$$
\begin{aligned}
u(t_2) = {} & Q_\alpha(t_2 - t_1)Q_\alpha(t_1)H(u) + Q_\alpha(t_2 - t_1)T_1\big(u\big(t_1^-\big)\big) \\
& + \int_0^{t_1} Q_\alpha(t_2 - t_1)Q_\alpha(t_1 - s)F(s, u(s))ds \\
& + \int_{t_1}^{t_2} Q_\alpha(t_2 - s)F(s, u(s))ds,
\end{aligned}
\tag{3.8}
$$

which implies that

$$
\begin{aligned}
u\big(t_2^+\big) = {} & Q_\alpha(t_2 - t_1)Q_\alpha(t_1)H(u) + Q_\alpha(t_2 - t_1)T_1\big(u\big(t_1^-\big)\big) \\
& + T_2\big(u\big(t_2^-\big)\big) + \int_0^{t_1} Q_\alpha(t_2 - t_1)Q_\alpha(t_1 - s)F(s, u(s))ds \\
& + \int_{t_1}^{t_2} Q_\alpha(t_2 - s)F(s, u(s))ds,
\end{aligned}
\tag{3.9}
$$

provided that $u(t_2^+) = u(t_2^-) + T_2(u(t_2^-))$. Then, again making use of Definition 2.1, we get for all $t \in I_2$,

$$
\begin{aligned}
u(t) &= Q_\alpha(t - t_2)u\big(t_2^+\big) + \int_{t_2}^{t} Q_\alpha(t - s)F(s, u(s))ds \\
&= Q_\alpha(t - t_2)Q_\alpha(t_2 - t_1)Q_\alpha(t_1)H(u) \\
&\quad + Q_\alpha(t - t_2)Q_\alpha(t_2 - t_1)T_1\big(u\big(t_1^-\big)\big) + Q_\alpha(t - t_2)T_2\big(u\big(t_2^-\big)\big) \\
&\quad + \int_0^{t_1} Q_\alpha(t - t_2)Q_\alpha(t_2 - t_1)Q_\alpha(t_1 - s)F(s, u(s))ds \\
&\quad + \int_{t_1}^{t_2} Q_\alpha(t - t_2)Q_\alpha(t_2 - s)F(s, u(s))ds \\
&\quad + \int_{t_2}^{t} Q_\alpha(t - s)F(s, u(s))ds, \\
&= \Phi_2^\alpha(t)H(u) + \int_{t_2}^{t} Q_\alpha(t - s)F(s, u(s))ds \\
&\quad + \sum_{1 \le j \le 2} \int_{t_{j-1}}^{t_j} \Phi_{2,j}^\alpha(t)Q_\alpha(t_j - s)F(s, u(s))ds \\
&\quad + \sum_{1 \le j \le 2} \Phi_{2,j}^\alpha(t)T_j\Big(u\Big(t_j^-\Big)\Big),
\end{aligned}
\tag{3.10}
$$

here $\Phi_2^{\alpha}(t)$ and $\Phi_{2,j}^{\alpha}(t)$ are given by (3.2) and (3.3) with $i = 2$, respectively. A continuation of the same process shows that for any $t \in I_i$ $(i = 1, \ldots, n)$, the assertion of theorem holds. □

In this work, we adopt the following concept of mild solution for the problem (1.3).

Definition 3.2. Let $1 < \alpha < 2$. Assume that A is a sectorial operator of type $\theta < 0$, $Q_\alpha(t)$ is a solution operator generated by A, and $\Phi_i^{\alpha}(t)$ and $\Phi_{i,j}^{\alpha}(t)$ are given by (3.2) and (3.3), respectively. A solution $u \in \mathrm{PC}([0,a];X)$ of the integral equation

$$\begin{aligned} u(t) = \Phi_i^{\alpha}(t)H(u) &+ \int_{t_i}^{t} Q_\alpha(t-s)F(s,u(s))ds \\ &+ \sum_{1\le j\le i}\int_{t_{j-1}}^{t_j} \Phi_{i,j}^{\alpha}(t)Q_\alpha\left(t_j - s\right)F(s,u(s))ds \\ &+ \sum_{1\le j\le i}\Phi_{i,j}^{\alpha}(t)T_j\left(u\left(t_j^-\right)\right), \quad t \in I_i, \end{aligned} \tag{3.11}$$

here $i = 0, 1, \ldots, n$, is called a mild solution of the Cauchy problem (1.3).

Remark 3.3. Note that if there is no discontinuity, that is, if $T_i(u(t_i^-)) = 0$, $i = 1, \ldots, n$, then Definition 2.1 is equivalent to Definition 3.2.

Now we present and prove our main results.

Theorem 3.4. *Let $1 < \alpha < 2$. Assume that A is a sectorial operator of type $\theta < 0$ and $Q_\alpha(t)$ is a solution operator generated by A. Suppose in addition that assumptions (H_1)–(H_3) are fulfilled. Then the Cauchy problem* (1.3) *admits at least one mild solution, provided*

$$\begin{gathered} C_\alpha^{n+1}\eta + aC_\alpha L_F + aC_\alpha^{n+1}L_F + C_\alpha^{n}\sum_{1\le j\le n}L_j < 1 \quad \text{if } C_\alpha \ge 1, \\ C_\alpha\eta + aC_\alpha L_F + aC_\alpha^{2}L_F + C_\alpha\sum_{1\le j\le n}L_j < 1 \quad \text{if } C_\alpha < 1. \end{gathered} \tag{3.12}$$

Proof. Consider the mapping $\Gamma^\alpha : \mathrm{PC}([0,a];X) \to \mathrm{PC}([0,a];X)$, which is defined for each $u \in \mathrm{PC}([0,a];X)$ by

$$\begin{aligned} (\Gamma^\alpha u)(t) = \Phi_i^{\alpha}(t)H(u) &+ \int_{t_i}^{t} Q_\alpha(t-s)F(s,u(s))ds \\ &+ \sum_{1\le j\le i}\int_{t_{j-1}}^{t_j} \Phi_{i,j}^{\alpha}(t)Q_\alpha\left(t_j - s\right)F(s,u(s))ds \\ &+ \sum_{1\le j\le i}\Phi_{i,j}^{\alpha}(t)T_j\left(u\left(t_j^-\right)\right), \quad t \in I_i,\ i = 0, 1, \ldots, n. \end{aligned} \tag{3.13}$$

Then it is clear that Γ^α is well defined.

To prove the theorem, it is sufficient to prove that Γ^α has a fixed point in $\mathrm{PC}([0,a];X)$. Put

$$W_r := \{v \in \mathrm{PC}([0,a];X); \|v(t)\| \leq r,\ \forall t \in [0,a]\} \tag{3.14}$$

for $r > 0$ as selected below.

We first show that there exists an integer $r > 0$ such that Γ^α maps W_r into W_r. For the case $t \in I_0$, by assumption (H_1) and the estimate (2.7), a straightforward calculation yields that

$$\begin{aligned}
&\|(\Gamma^\alpha u)(t)\| \\
&\quad \leq \|Q_\alpha(t)H(u)\| + \int_0^t \|Q_\alpha(t-s)F(s,u(s))\|ds \\
&\quad \leq \|Q_\alpha(t)\|_{\mathcal{L}(X)}\|H(u)\| + \int_0^t \|Q_\alpha(t-s)\|_{\mathcal{L}(X)}\|F(s,0)\|ds \\
&\qquad + \int_0^t \|Q_\alpha(t-s)\|_{\mathcal{L}(X)}\|F(s,u(s)) - F(s,0)\|ds \\
&\quad \leq C_\alpha\|H(u)\| + C_\alpha L_F \int_0^t \|u(s)\|ds + t_1 C_\alpha \sup_{s\in I_0}\|F(s,0)\|.
\end{aligned} \tag{3.15}$$

We claim that there exists an integer $r > 0$ such that $\|(\Gamma^\alpha u)(t)\| \leq r$ provided that $u \in W_r$. In fact, if this is not the case, then for each $N > 0$, there would exist $u \in W_N$ and $t_N \in I_0$ such that $\|(\Gamma^\alpha u_N)(t_N)\| > N$. Thus, by (3.15) and assumption (H_2) we obtain

$$N < \|(\Gamma^\alpha u_N)(t_N)\| \leq C_\alpha \Phi(N) + t_1 N C_\alpha L_F + t_1 C_\alpha \sup_{s\in I_0}\|F(s,0)\|. \tag{3.16}$$

Dividing on both sides by N and taking the lower limit as $N \to +\infty$, we get

$$C_\alpha \eta + t_1 C_\alpha L_F \geq 1, \tag{3.17}$$

which contradicts (3.12).

Since the interval $[0,a]$ is divided into finite subintervals by t_i, $i = 1,\ldots,n$, we only need to prove that for a fixed i $(\in \{1,\ldots,n\})$,

$$(\Gamma_i^\alpha u)(t) := (\Gamma^\alpha u)(t)|_{t\in I_i} \tag{3.18}$$

maps W_r into W_r, here $r > 0$ is a positive number yet to be determined, as the cases for other subintervals are the same.

From the Hypotheses (H_1)–(H_3), we infer for any $u \in W_r$,

$$\begin{aligned}
\left\| \left(\Gamma_i^{\alpha} u\right)(t) \right\| \leq{}& \left\| \Phi_i^{\alpha}(t) H(u) \right\| + \sum_{1 \leq j \leq i} \int_{t_{j-1}}^{t_j} \left\| \Phi_{i,j}^{\alpha}(t) Q_{\alpha}\left(t_j - s\right) F(s, u(s)) \right\| ds \\
&+ \int_{t_i}^{t} \| Q_{\alpha}(t-s) F(s, u(s)) \| ds + \sum_{1 \leq j \leq i} \left\| \Phi_{i,j}^{\alpha}(t) T_j\left(u\left(t_j^-\right)\right) \right\| \\
\leq{}& \left\| \Phi_i^{\alpha}(t) \right\|_{\mathcal{L}(X)} \| H(u) \| + \int_{t_i}^{t} \| Q_{\alpha}(t-s) \|_{\mathcal{L}(X)} \| F(s, 0) \| ds \\
&+ \int_{t_i}^{t} \| Q_{\alpha}(t-s) \|_{\mathcal{L}(X)} \| F(s, u(s)) - F(s, 0) \| ds \\
&+ \sum_{1 \leq j \leq i} \int_{t_{j-1}}^{t_j} \left\| \Phi_{i,j}^{\alpha}(t) Q_{\alpha}\left(t_j - s\right) \right\|_{\mathcal{L}(X)} \| F(s, u(s)) - F(s, 0) \| ds \\
&+ \sum_{1 \leq j \leq i} \int_{t_{j-1}}^{t_j} \left\| \Phi_{i,j}^{\alpha}(t) Q_{\alpha}\left(t_j - s\right) \right\|_{\mathcal{L}(X)} \| F(s, 0) \| ds \\
&+ \sum_{1 \leq j \leq i} \left\| \Phi_{i,j}^{\alpha}(t) T_j(0) \right\| + \sum_{1 \leq j \leq i} \left\| \Phi_{i,j}^{\alpha}(t) \right\|_{\mathcal{L}(X)} \left\| T_j\left(u\left(t_j^-\right)\right) - T_j(0) \right\| \\
\leq{}& C_{\alpha}^{i+1} \Phi(r) + (t - t_i) C_{\alpha} \sup_{\tau \in I_i} \| F(\tau, 0) \| + (t - t_i) C_{\alpha} L_F r \\
&+ t_i C_{\alpha}^{i+1} L_F r + C_{\alpha}^{i} \sum_{1 \leq j \leq i} \left\| T_j(0) \right\| + C_{\alpha}^{i} r \sum_{1 \leq j \leq i} L_j \\
&+ C_{\alpha}^{i+1} \sum_{1 \leq j \leq i} \left(t_j - t_{j-1}\right) \sup_{\tau \in \left[t_{j-1}, t_j\right]} \| F(\tau, 0) \|.
\end{aligned} \tag{3.19}$$

Now, an application of the same idea with above discussion yields that there exists a $r > 0$ such that $\|(\Gamma_i^{\alpha} u)(t)\| \leq r$. Indeed, if this is not the case, then we would deduce that

$$C_{\alpha}^{i+1} \eta + (t - t_i) C_{\alpha} L_F + t_i C_{\alpha}^{i+1} L_F + C_{\alpha}^{i} \sum_{1 \leq j \leq i} L_j \geq 1. \tag{3.20}$$

This is a contradiction to (3.12). Thus, we prove that there exists an integer $r > 0$ such that $\Gamma^{\alpha}(W_r) \subset W_r$.

For $i = 0, 1, \ldots, n$, we decompose the mapping $\Gamma^{\alpha} = \Gamma_1^{\alpha} + \Gamma_2^{\alpha}$ as follows:

$$\begin{aligned}
\left(\Gamma_1^{\alpha} u\right)(t) &= \Phi_i^{\alpha}(t) H(u), \quad t \in I_i, \\
\left(\Gamma_2^{\alpha} u\right)(t) &= \int_{t_i}^{t} Q_{\alpha}(t-s) F(s, u(s)) ds + \sum_{1 \leq j \leq i} \Phi_{i,j}^{\alpha}(t) T_j\left(u\left(t_j^-\right)\right) \\
&\quad + \sum_{1 \leq j \leq i} \int_{t_{j-1}}^{t_j} \Phi_{i,j}^{\alpha}(t) Q_{\alpha}\left(t_j - s\right) F(s, u(s)) ds \\
&:= \int_{t_i}^{t} Q_{\alpha}(t-s) F(s, u(s)) ds + \left(\Gamma_T^{\alpha} u\right)(t) + \left(\Gamma_{\mathrm{In}}^{\alpha} u\right)(t), \quad t \in I_i.
\end{aligned} \tag{3.21}$$

Next, we show that for each i $(i = 0, 1, \ldots, n)$, Γ_1^α is completely continuous, while $(\Gamma_2^\alpha u)(t)|_{t\in I_i}$ is a contraction. In fact, it follows from assumption (H_2) and the estimate (2.7) that $\Gamma_1^\alpha|_{I_i}$, $i = 0, 1, \ldots, n$ is completely continuous. Note also that

$$(\Gamma_{\mathrm{In}}^\alpha u)(t) = \begin{cases} 0, & t \in I_0, \\ \displaystyle\int_0^{t_1} Q_\alpha(t-t_1)Q_\alpha(t_1 - s)F(s, u(s))ds, & t \in I_1, \\ \cdots \\ \displaystyle\sum_{1\le j\le n}\int_{t_{j-1}}^{t_j} \Phi_{n,j}^\alpha(t)Q_\alpha(t_j - s)F(s,u(s))ds, & t \in I_n, \end{cases}$$
$$(\Gamma_T^\alpha u)(t) = \begin{cases} 0, & t \in I_0, \\ Q_\alpha(t-t_1)T_1(u(t_1^-)), & t \in I_1, \\ \cdots \\ \displaystyle\sum_{1\le j\le n} \Phi_{n,j}^\alpha(t)T_j\left(u\left(t_j^-\right)\right), & t \in I_n. \end{cases} \tag{3.22}$$

For the case $i = 0$, it is clear that the conclusion holds in view of (3.12). For $t \in I_i$ $(i = 1, \ldots, n)$, by (H_1), (H_3) and (2.7) we get

$$\begin{aligned} &\left\|(\Gamma_2^\alpha u)(t) - (\Gamma_2^\alpha w)(t)\right\| \\ &\quad\le \int_{t_i}^t \|Q_\alpha(t-s)(F(s,u(s)) - F(s,w(s)))\|ds \\ &\qquad + \sum_{1\le j\le i}\Phi_{i,j}^\alpha(t)\left(T_j\left(u\left(t_j^-\right)\right) - T_j\left(w\left(t_j^-\right)\right)\right) \\ &\qquad + \sum_{1\le j\le i}\int_{t_{j-1}}^{t_j}\left\|\Phi_{i,j}^\alpha(t)Q_\alpha(t_j - s)(F(s,u(s)) - F(s,w(s)))\right\|ds \\ &\quad\le C_\alpha L_F\int_{t_i}^t \|u(s) - w(s)\|ds + C_\alpha^i\sum_{1\le j\le i}L_j\left\|u(t_j) - w(t_j)\right\| \\ &\qquad + C_\alpha^{i+1}L_F\sum_{1\le j\le i}\int_{t_{j-1}}^{t_j}\|u(s) - \mathrm{w}(s)\|ds \\ &\quad\le (t-t_i)C_\alpha L_F\|u(s) - w(s)\|_{\mathrm{PC}} + t_iC_\alpha^{i+1}L_F\|u(s) - w(s)\|_{\mathrm{PC}} \\ &\qquad + C_\alpha^i\|u(s) - w(s)\|_{\mathrm{PC}}\sum_{1\le j\le i}L_j \\ &\quad\le \left((t-t_i)C_\alpha L_F + t_iC_\alpha^{i+1}L_F + C_\alpha^i\sum_{1\le j\le i}L_j\right)\|u(s) - w(s)\|_{\mathrm{PC}}, \end{aligned} \tag{3.23}$$

provided that $u, w \in W_r$. Hence, we deduce that

$$\left\|\Gamma_2^\alpha u - \Gamma_2^\alpha w\right\|_{\mathrm{PC}} \leq \left((t-t_i)C_\alpha L_F + t_i C_\alpha^{i+1} L_F + C_\alpha^i \sum_{1\leq j\leq i} L_j\right)\|u-w\|_{\mathrm{PC}}, \tag{3.24}$$

which means that Γ_2^α is a contraction due to (3.12).

Thus, $\Gamma^\alpha = \Gamma_1^\alpha + \Gamma_2^\alpha$ is a condensing map on W_r. Then, it follows from Lemma 2.2 that the Cauchy problem (1.3) admits at least one mild solution. This completes the proof. □

Theorem 3.5. *Let* $1 < \alpha < 2$. *Assume that* A *is a sectorial operator of type* $\theta < 0$, $Q_\alpha(t)$ *is a solution operator generated by* A, *and the Hypotheses* (H_1), (H_2), (H_3') *are satisfied. Then the Cauchy problem* (1.3) *admits at least one mild solution, provided*

$$\begin{gathered} C_\alpha^{n+1}\eta + aC_\alpha L_F + aC_\alpha^{n+1} L_F + C_\alpha^n \sum_{1\leq j\leq n} \lambda_j < 1 \quad \text{if } C_\alpha \geq 1, \\ C_\alpha\eta + aC_\alpha L_F + aC_\alpha^2 L_F + C_\alpha \sum_{1\leq j\leq n} \lambda_j < 1 \quad \text{if } C_\alpha < 1. \end{gathered} \tag{3.25}$$

Proof. Assume that the map $\Gamma^\alpha : \mathrm{PC}([0,a];X) \to \mathrm{PC}([0,a];X)$ and the set W_r are defined the same as in Theorem 3.4. First we claim that there exists an positive number $r > 0$ such that $\Gamma^\alpha(W_r) \subset W_r$. For the case $t \in I_0$, the proof of the assertion follows from Theorem 3.4. For the case $t \in I_i$ $(i = 1, \ldots, n)$, if the conclusion is not true, then for each positive integer r, there would exist $u_r(\cdot) \in W_r$ and $t_r \in I_i$ such that $\|(\Gamma_i^\alpha u_r)(t_r)\| > r$ with $(\Gamma_i^\alpha u)(t) := (\Gamma^\alpha u)(t)|_{t\in I_i}$, where t_r denotes t depending upon r. Thus, by assumptions (H_1), (H_2), (H_3'), we have

$$\begin{aligned} r &< \left\|(\Gamma_i^\alpha u_r)(t_r)\right\| \leq \left\|\Phi_i^\alpha(t_r)H(u_r)\right\| \\ &\quad + \int_{t_i}^{t_r} \|Q_\alpha(t_r - s)F(s, u_r(s))\| ds \\ &\quad + \sum_{1\leq j\leq i} \int_{t_{j-1}}^{t_j} \|\Phi_{i,j}^\alpha(t_r) Q_\alpha(t_j - s)F(s, u_r(s))\| ds \\ &\quad + \sum_{1\leq j\leq i} \|\Phi_{i,j}^\alpha(t_r) T_j\left(u_r\left(t_j^-\right)\right)\| \\ &\leq C_\alpha^{i+1}\Phi(r) + (t_r - t_i)C_\alpha \sup_{\tau\in I_i}\|F(\tau,\theta)\| + (t_r - t_i)C_\alpha L_F r \\ &\quad + t_i C_\alpha^{i+1} L_F r + C_\alpha^{i+1} \sum_{1\leq j\leq i}\left(t_j - t_{j-1}\right) \sup_{\tau\in[t_{j-1},t_j]} \|F(\tau,\theta)\| \\ &\quad + C_\alpha^i \sum_{1\leq j\leq i} \Upsilon_j(r). \end{aligned} \tag{3.26}$$

Dividing on both sides by r and taking the lower limit as $r \to +\infty$, we have

$$C_\alpha^{i+1}\eta + aC_\alpha L_F + aC_\alpha^{i+1}L_F + C_\alpha^i \sum_{1\le j\le i}\lambda_j \ge 1. \tag{3.27}$$

This is a contradiction to (3.25).

For $i = 0,1,\dots,n$, decompose the mapping $\Gamma^\alpha = \Gamma_1^\alpha + \Gamma_2^\alpha$ as follows:

$$\begin{aligned}(\Gamma_1^\alpha u)(t) &= \Phi_i^\alpha(t)H(u) + \sum_{1\le j\le i}\Phi_{i,j}^\alpha(t)T_j\left(u\left(t_j^-\right)\right), \quad t\in I_i\\ (\Gamma_2^\alpha u)(t) &= \int_{t_i}^t Q_\alpha(t-s)F(s,u(s))ds + \sum_{1\le j\le i}\int_{t_{j-1}}^{t_j}\Phi_{i,j}^\alpha(t)Q_\alpha(t_j-s)F(s,u(s))ds.\end{aligned} \tag{3.28}$$

Next, we will verify that for each $t\in I_i$ $(i=0,1,\dots,n)$, Γ_1^α is a completely continuous operator, while, Γ_2^α is a contraction. Obviously, by assumptions (H_2), (H_3'), it easily seen that Γ_1^α is a completely continuous operator. Moreover, by a similar proof with that in Theorem 3.4, we can prove that Γ_2^α is a contraction.

As a consequence of the above discussion and Lemma 2.2, we can conclude that the problem (1.3) admits at least one mild solution. The proof is completed. □

Theorem 3.6. *Let $1<\alpha<2$. Assume that A is a sectorial operator of type $\theta<0$ and $Q_\alpha(t)$ is a solution operator generated by A. Then, under assumptions (H_1'), (H_2'), (H_3), the Cauchy problem* (1.3) *has a unique mild solution, provided*

$$\begin{aligned}C_\alpha^{n+1}L_H + \left(C_\alpha + C_\alpha^{n+1}\right)\int_0^a \rho(s)ds + C_\alpha^n\sum_{1\le j\le n}L_j < 1 \quad \text{if } C_\alpha \ge 1,\\ C_\alpha L_H + \left(C_\alpha + C_\alpha^2\right)\int_0^a \rho(s)ds + C_\alpha\sum_{1\le j\le n}L_j < 1 \quad \text{if } C_\alpha < 1.\end{aligned} \tag{3.29}$$

Proof. Assume that the map $\Gamma^\alpha : \mathrm{PC}([0,a];X) \to \mathrm{PC}([0,a];X)$ is defined the same as in Theorem 3.4. Now, we prove that Γ^α is a contraction. Take any $u,w\in \mathrm{PC}([0,a];X)$. For the case $t\in I_0$, the conclusion follows from assumptions (H_1'), (H_2'), and (3.29). For $t\in I_i$ $(i=1,\dots,n)$, a direct calculation yields

$$\begin{aligned}&\|(\Gamma^\alpha u)(t) - (\Gamma^\alpha w)(t)\|\\ &\quad\le \left\|\Phi_i^\alpha(t)(H(u)-H(w))\right\|\\ &\qquad+\int_{t_i}^t \|Q_\alpha(t-s)(F(s,u(s))-F(s,w(s)))\|ds\\ &\qquad+\sum_{1\le j\le i}\int_{t_{j-1}}^{t_j}\left\|\Phi_{i,j}^\alpha(t)Q_\alpha(t_j-s)(F(s,u(s))-F(s,w(s)))\right\|ds\\ &\qquad+\sum_{1\le j\le i}\left\|\Phi_{i,j}^\alpha(t)\left(T_j\left(u\left(t_j^-\right)\right)-T_j\left(w\left(t_j^-\right)\right)\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq C_\alpha^{i+1} L_H \|u-w\|_{\mathrm{PC}} + C_\alpha \|u-w\|_{\mathrm{PC}} \int_{t_i}^{t} \rho(s)ds \\
&\quad + C_\alpha^{i+1} \|u-w\|_{\mathrm{PC}} \int_0^t \rho(s)ds + C_\alpha^i \sum_{1\leq j\leq i} L_j \left\| u\left(t_j^-\right) - w\left(t_j^-\right) \right\| \\
&\leq \left(C_\alpha^{i+1} L_H + C_\alpha \int_{t_i}^{t} \rho(s)ds + C_\alpha^{i+1} \int_0^t \rho(s)ds + C_\alpha^i \sum_{1\leq j\leq i} L_j \right) \|u-w\|_{\mathrm{PC}} \\
&\leq \left(C_\alpha^{i+1} L_H + \left(C_\alpha + C_\alpha^{i+1}\right) \int_0^a \rho(s)ds + C_\alpha^i \sum_{1\leq j\leq i} L_j \right) \|u-w\|_{\mathrm{PC}}.
\end{aligned}
\tag{3.30}
$$

in view of assumptions (H_1'), (H_2'), (H_3). Hence, we deduce that

$$
\begin{aligned}
&\|\Gamma^\alpha u - \Gamma^\alpha w\|_{\mathrm{PC}} \\
&\quad \leq \left(C_\alpha^{i+1} L_H + \left(C_\alpha + C_\alpha^{i+1}\right) \int_0^a \rho(s)ds + C_\alpha^i \sum_{1\leq j\leq i} L_j \right) \|u-w\|_{\mathrm{PC}},
\end{aligned}
\tag{3.31}
$$

which implies Γ^α is a contractive mapping on $\mathrm{PC}([0,a];X)$ due to (3.29). Thus Γ^α has a unique fixed point $u \in \mathrm{PC}([0,a];X)$, this means that u is a mild solution of (1.3). This completes the proof of the theorem. □

4. Example

In this section, we present an example to illustrate the abstract results of this paper, which do not aim at generality but indicate how our theorems can be applied to concrete problems.

Consider the BVP of partial differential equation in the form

$$
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} L_x u(s,x)ds = \frac{|u(t,x)|}{C(1+|u(t,x)|)}, \quad 0 \leq t \leq a,\ t \neq t_i,\ 0 \leq x \leq \pi, \\
&u(t,0) = u(t,\pi) = 0, \quad 0 \leq t \leq a, \\
&u(0,x) = \frac{1}{C}\left|u(t_0', x)\right|, \quad 0 \leq x \leq \pi, \\
&u(t_i^+, x) = u(t_i^-, x) + \frac{|u(t_i,x)|}{(i+nC) + t_i|u(t_i,x)|}, \quad i = 1, \ldots, n,
\end{aligned}
\tag{4.1}
$$

where $1 < \alpha < 2$, t_0' is a constant in $(0, a)$, $C > 0$ is a constant yet to be determined, L_x stands for the operator with respect to the spatial variable x which is given by

$$L_x = \frac{\partial^2}{\partial x^2} - v \quad (v > 0). \tag{4.2}$$

In what follows we consider the space $X = L^2[0, \pi]$ with norm $\|\cdot\|_2$ and the operator $A := L_x : D(A) \subset X \to X$ with domain

$$\left\{u \in \mathrm{X};\ u, u' \text{ are absolutely continuous, } u'' \in \mathrm{X}, \text{ and } u(0) = u(\pi) = 0\right\}. \tag{4.3}$$

Clearly A is densely defined in X and is sectorial of type $\theta = -\nu < 0$. Hence A is a generator of a solution operator satisfying the estimate (2.7) on X. Here, without lost of generality, we take $C_\alpha \geq 1$.

Set

$$\begin{gathered}
u(t)(x) = u(t, x), \\
F(t, u(t))(x) = \frac{|u(t, x)|}{C(1 + |u(t, x)|)}, \\
H(u)(x) = \frac{1}{C}\left|u(t_0', x)\right|, \\
T_i(u(t_i))(x) = \frac{|u(t_i, x)|}{(i + nC) + t_i|u(t_i, x)|}, \quad i = 1, \ldots, n.
\end{gathered} \tag{4.4}$$

Then we have

$$\begin{gathered}
\|F(t, u(t)) - F(t, v(t))\|_2 \leq \frac{1}{C}\|u(t) - v(t)\|_2, \quad 0 \leq t \leq a, \\
\|H(u) - H(v)\|_2 \leq \frac{1}{C}\|u - v\|_2, \\
\|T_i(u) - T_i(v)\|_2 \leq \frac{1}{i + nC}\|u - v\|_2, \quad i = 1, \ldots, n.
\end{gathered} \tag{4.5}$$

Note that the problem (4.1) also can be reformulated as the abstract problem (1.3), and due to (4.5), it is not difficult to see that assumptions (H_1'), (H_2'), and (H_3) hold with

$$\rho(t) = \frac{1}{C} \quad (t \in [0, \mathrm{a}]), \qquad L_H = \frac{1}{C}, \qquad L_i = \frac{1}{i + nC}, \qquad i = 1, \ldots, n, \tag{4.6}$$

which implies that one can choose large enough C such that the first inequality of (3.29) is satisfied. Hence, according to Theorem 3.6, the Cauchy problem (4.1) has a unique mild solution.

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